Self-Assessment Questions for the specialized Master of Science Quantitative Finance, jointly offered by UZH & ETH

> I advise my students to listen carefully the moment they decide to take no more mathematics courses. They might be able to hear the sound of closing doors.
>
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This document serves to indicate the kind of knowledge we expect incoming students to have, and are the kinds of questions that we ask in the interview. Its purpose is to assist you in self-selection: If you are comfortable and competent with, say, 75% of the questions, we would encourage you to apply, though note that this is a necessary but not sufficient condition, and all the other requirements need to be met. Application fees are not returned, so please self-assess adequately.

Contents

1 Calculus Facts You Should Know **2**

2 Basic Calculus Questions **3**

3 Basic Probability Questions **5**

4 Stochastic Processes **6**

5 Asset Pricing / Financial Economics Questions **6**

1 Calculus Facts You Should Know

- Recall the *Fundamental Theorem of Calculus* (FTC): Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Then:

  1. Define $F : [a, b] \to \mathbb{R}$ by
  $$F(x) := \int_a^x f(t)\,\mathrm{d}t.$$
  If f is continuous at $c \in [a, b]$, then F is differentiable at c and $F'(c) = f(c)$. Consequently, if f is continuous on $[a, b]$, then F is differentiable on $[a, b]$ and $F' = f$.

  2. If f is differentiable and f' is integrable on $[a, b]$, then
  $$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a).$$

- Recall the *Taylor formula for function f around point a*: If $I \subseteq \mathbb{R}$ is an interval, a is any point in I, and function $f : I \to \mathbb{R}$ is such that $f', f'', \ldots, f^{(n)}$ exist on I and $f^{(n+1)}$ exists at every interior point of I, then for every $x \in I \setminus \{a\}$, there is a point c between a and x such that

$$\begin{aligned} f(x) &= P_n(x) + R_n(x) \\ &= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}, \end{aligned} \qquad (1)$$

where $P_n(x)$ is the nth *Taylor polynomial of f around a*, and $R_n(x)$ is the *remainder of order n*. If f is infinitely differentiable, then $P_n(x)$ as $n \to \infty$ (and $R_n(x) = 0$) is called the *Taylor series* of f around a. (If $a = 0$, it is sometimes called the *Maclaurin series*).

2 Basic Calculus Questions

1. (Taylor series for the exponential function)

   (a) Using (1), show that the Taylor series of $\exp(x) = \mathrm{e}^x$ around the point zero is given by
   $$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{2}$$

   (b) Using (2), compute (what turns out to be the expectation of a Poisson random variable)
   $$\sum_{y=0}^{\infty} \frac{y\mathrm{e}^{-\lambda}\lambda^y}{y!}, \quad \lambda > 0.$$

2. (Taylor series for log)

   (a) For $a = 0$ and $I = (-1, 1)$, compute the Taylor series of $f = \log(1 + x)$ for $x \in I$ to show that
   $$\log(1 + x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}, \quad x \in (-1, 1). \tag{3}$$
   It can be shown[1] that we can take $I = (-1, 1]$, so that
   $$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots . \tag{4}$$
   See also question **??** below.

   (b) Compute the Taylor series of $g(x) = \log(x)$ about the point $a = 1$, and confirm (4).

3. (Understanding of existence of integral, Cauchy principle value)

   (a) Let $f(x) = x/(1 + x^2)$. For $K > 0$ derive
   $$\int_0^K \frac{x}{1+x^2}\,\mathrm{d}x \quad \text{and} \quad \int_{-K}^0 \frac{x}{1+x^2}\,\mathrm{d}x, \quad \text{and thus} \quad \int_{-K}^K \frac{x}{1+x^2}\,\mathrm{d}x.$$

   (b) What can you conclude about
   $$\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,\mathrm{d}x\ ?$$

4. State the definition the binomial coefficient (i.e., "n choose k"). Explain in words why, and algebraically prove, that
   $$\binom{n}{k} = \binom{n}{n-k}.$$

5. (Binomial Theorem)

[1] See, e.g., Ghorpade and Limaye (2018, p. 389) or, via Abel's theorem, Duren (2012, p. 77).

(a) State the binomial theorem. (Don't forget to indicate the sets on which the variables you define are defined.)

(b) State the basic calculus definition of the first derivative of a function (the Newton quotient) in terms of a limit.

(c) Use the definition of derivative to derive the derivative of $f(x) = x^n$, where $n \in \{1, 2, 3, \dots\}$ is a natural number.

6. For a geometric series $\{ar^k\}_{k=0}^{\infty}$, $a \in \mathbb{R}$ and $|r| < 1$, prove that the infinite sum has the following closed form

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

7. Evaluate the integral

$$\int_{-\infty}^{\infty} x^2 \mathrm{e}^{-2x^3} \, \mathrm{d}x.$$

8. The definition of the gamma function is

$$\Gamma(a) = \int_0^{\infty} x^{a-1} \mathrm{e}^{-x} \, \mathrm{d}x, \ a \in \mathbb{R}_{>0}.$$

It is well known that there is no closed-form antiderivative expression. Show that, for $a > 1$,

$$\Gamma(a) = (a-1)\Gamma(a-1).$$

3 Basic Probability Questions

1. Let $X_1 \sim \text{Bin}(n_1, p)$ independent of $X_2 \sim \text{Bin}(n_2, p)$, under the usual restriction of the parameter range, and let $S = X_1 + X_2$. State the distribution of S and its expected value and variance.

2. Let $X_1 \sim \text{Bin}(n_1, p_1)$ independent of $X_2 \sim \text{Bin}(n_2, p_2)$, under the usual restriction of the parameter range, and with $p_1 \neq p_2$. Let $S = X_1 + X_2$. Write a computable formula for the cdf of S.

3. Assume you and your spouse decide to have children until you amass 3 boys. Ignore any obvious biological constraints! Further assume p is the probability of getting a boy on each trial. Compute the expected number of children associated with this family planning strategy.

4. Let $X \sim \text{Norm}(0, \sigma^2)$. Without calculation, state $\text{E}[X^2]$, $\text{E}[X^3]$, and $\text{E}[X^4]$.

5. Given an unlimited set of IID standard Gaussian random variables, i.e., $\text{Norm}(0, 1)$, explain how you would simulate: (i) a realization from a χ^2_n and (ii) a Student's t realization with n degrees of freedom.

6. Recall that a probability measure is a set function that assigns a real number $\Pr(A)$ to each event $A \in \mathcal{A}$, and has 3 requirements. State them, and describe what is meant here by $\mathcal{A}$.

7. Given the characteristic function of a continuous univariate random variable, explain in words how you can compute the pdf.

8. In words, explain the (weak) law of large numbers, and the central limit theorem, and how they differ.

9. State 5 common univariate discrete distributions. (You do not need to write out the mass functions.)

10. State a (common) distribution whose expectation does not exist.

11. State the pdf and cdf of an exponential random variable with rate λ. Given a set of IID exponential random variables, how is their sum distributed?

12. (one question on linear algebra) For a real symmetric matrix, state relationships between the eigenvalues, the trace, and the determinant of the matrix.

4 Stochastic Processes

1. Consider a finite probability space $(\Omega, \mathcal{F}, P)$ with $\mathcal{F}$ being the power set and a probability measure P such that $P(\omega) > 0$ for $\omega \in \Omega$. Define a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ and define the conditional expectation $E[X|\mathcal{G}]$ for a random variable $X : \Omega \to \mathbb{R}$. Calculate the conditional expectation in case that $\mathcal{G} = \{\emptyset, \Omega, A, \Omega \setminus A\}$ for $P(A)P(\Omega \setminus A) > 0$.

2. Give an argument why the conditional expectation is the best estimate (best with respect to what?) of a random variable X given information encoded in $\mathcal{G}$.

3. Consider a random walk on the grid of integers changing at each time step either by X_i taking values $+1$ or -1 with probability $1/2$ in an independent manner and starting at 0. Define the value of the random walk by $S_n = \sum_{i=1}^{n} X_i$. Prove that $E[S_n|\sigma(X_1, \ldots, X_{n-1}] = S_{n-1}$ for $n \geq 1$, where $\sigma(X_1, \ldots, X_{n-1})$ is the σ-algebra generated by $X_1, \ldots, X_{n-1}$. Why is this result reasonable?

4. How does the result from the previous question change when probabilities are instead of $1/2$ and $1/2$ given by $2/3$ for $+1$ and $1/3$ for -1?

5. Consider a random walk from Question 3 and consider the first time τ when the random walk hits the level 5. Calculate $E[S_\tau]$ and give an intuitive argument why this is reasonable.

6. Describe the reflection principle for discrete random walks in words and in formal terms.

7. Define a discrete time martingale. Is (S_n) from Question 3 a martingale?

8. Consider a sequence of i.i.d. random variables (X_i) with mean μ and finite variance $\sigma^2 > 0$. Write down in terms of formulas the strong law of large numbers and the central limit theorem for this sequence. Put it in relation to the above description in words.

9. Give an example when the law of large numbers is failing, and what it means.

10. Describe the difference of uncorrelated and independent random variables. Does one need for the above formulation of the law of large number the random variables to be independent or is it enough when they are uncorrelated.

11. Describe the Monte Carlo algorithm to calculate the integral $\int_{[0,1]^d} f(x)dx$ in words and pseudocode (with precise conditions). How much does the result of the algorithm deviate from the true value when n samples are used? Is this a deterministic or a probabilistic statement?

12. Does the convergence rate of the Monte Carlo estimator deteriorate with d when $\int_{[0,1]^d} f(x)^2dx$ remains constant? Why is this useful?

5 Asset Pricing / Financial Economics Questions

1. State the definitions of "simple returns" and "percentage log returns", and explain in words their relationship.

2. State some so-called "stylized facts" of daily financial asset returns.